

Family of closed convex sets covering faces of the simplex

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Abstract

Let $S = \text{co}\{a_1, a_2, \dots, a_{n+1}\}$ be the simplex spanned by the independent points $a_i \in \mathbb{R}^n$, $i = 1, 2, \dots, n+1$, and denote by S^i its face $\text{co}\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}\}$. If a family of $n+1$ closed convex sets A^1, A^2, \dots, A^{n+1} in S has the property $S^i \subset A^i$ for each i , then there exists a point v in S which is at the distance $\varepsilon_0 \geq 0$ to every set A^i . If $\varepsilon_0 > 0$, then the point v with this property is unique.

1 Introduction

There is classical literature of the combinatorial and algebraic topology considering the problem of the covering of a simplex. As the first and most famous result we mention Sperner's Lemma [11]. A dual result to the Lemma of Sperner is the Knaster-Kuratowski-Mazurkiewicz Theorem [4]. These results give raise to extensive investigations and applications, among which we mention the results in [2] and [10]. Some covering problems with closed convex sets benefit substantially from the above mentioned issues, as is reflected in [1], [3] and [2]. As far as we know there was no attempt in a more direct, more geometric approach in the convex case, although this special context allows to obtain specific results. In our recent paper [9] on families of convex sets we also follow the line of using essentially the classical covering theorems with closed sets, but as we shall show, a method developed there allows to prove an extended convex variant of Sperner's lemma without combinatorial reasonings. In this note we are doing this. Thus all the results concerning convex sets in [1], [3], [2] and [9] can be obtained by using our Theorem 1 in place of Sperner's lemma.

Our note continues the line of our earlier investigations in [6], [7], [8] and [5] about the existence of equally spaced points to some families of compact and convex sets in the Euclidean and Minkowski spaces.

2 Main results

Let $S = \text{co}\{a_1, a_2, \dots, a_{n+1}\}$ be the simplex spanned by the independent points $a_i \in \mathbb{R}^n$, $i \in N = \{1, 2, \dots, n+1\}$, and denote by S^i its face $\text{co}\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}\}$. Our main result is as follows:

Theorem 1 *If a family of $n+1$ closed convex sets A^1, A^2, \dots, A^{n+1} in S has the property $S^i \subset A^i$ for each i , then there exists a point v in S and an $\varepsilon_0 \geq 0$ such that v is at the distance ε_0 to every set A^i . If $\varepsilon_0 > 0$, then the point v with this property is unique.*

A family $\{B^\alpha : \alpha \in I\}$ is said to be a *face covering* for S if each B^α contains some face S^i .

The unicity of the equally spaced point in the first part of Theorem 1 concludes

Corollary 1 *If $\{B^\alpha : \alpha \in I\}$, $\text{card } I \geq n+2$ is a face covering family of closed convex sets in S such that every S^i , $i = 1, 2, \dots, n+1$ is contained in some of its sets and each subfamily with $n+2$ members possesses an equally spaced point in S of positive distance, then the whole family possesses such a point.*

Corollary 2 *The face covering family $\mathcal{B} = \{B^\alpha : \alpha \in I\}$ of closed convex sets in S has a nonempty intersection if and only if every subfamily $B^{\alpha_1}, B^{\alpha_2}, \dots, B^{\alpha_{n+1}}$ of it, which covers the boundary of S , covers also S .*

3 The proofs

Let us consider $N = \{1, 2, \dots, n+1\}$ and a family $\mathcal{H} = \{A^1, A^2, \dots, A^{n+1}\}$ of closed convex sets in S . Then \mathcal{H} will be called an *\mathcal{H} -family*, if $S^i \subset A^i$, $i \in N$ and $S \setminus \bigcup_{i \in N} A^i \neq \emptyset$.

We prove Theorem 1 by using two auxiliary lemmas. The first of them is

Lemma 1 *Consider the \mathcal{H} -family $\{A^1, A^2, \dots, A^{n+1}\}$. Let A_ε^i be the $\varepsilon > 0$ -hull of the set A^i in S , i. e., the set of points in S with the distance $\leq \varepsilon$ from the set A^i . Then*

1. *There exists an $\varepsilon_0 > 0$ such that:*
 - (i) $\{A_\varepsilon^i : i \in N\}$ is a \mathcal{H} -family for $\varepsilon < \varepsilon_0$,
 - (ii) $B_\varepsilon = \bigcap_{i \in N} A_\varepsilon^i \neq \emptyset$ for $\varepsilon \geq \varepsilon_0$.

2. B_{ε_0} reduces to a single point v , and ε_0 is the common distance of v to A^i .
3. v is the single point which is equally spaced to the sets A^i .
4. There holds the relation

$$\varepsilon_0 = \sup_{u \in S} \inf_{i \in N} d(u, A^i),$$

where $d(a, A)$ denotes the distance of the point a from the set A .

5. If $\{D^1, D^2, \dots, D^{n+1}\}$ is another \mathcal{H} -family with $A^i \subset D^i$, $i = 1, 2, \dots, n+1$ and ε_1 is the common distance of the equally spaced point from D^i , then $\varepsilon_1 \leq \varepsilon_0$.

Proof.

Our proof consists in fact from gathering some considerations in the proofs in [9].

Let A_ε^i be the closed $\varepsilon \geq 0$ -hull of the set A^i in S , i. e., the set of points in S with the distance $\leq \varepsilon$ from the set A^i . Denote $B_\varepsilon = \cap_{i \in N} A_\varepsilon^i$. Then

$$\Omega = \{\varepsilon : B_\varepsilon \neq \emptyset\}$$

is obviously not empty and $\varepsilon_0 = \inf \Omega$ is well defined. Indeed, since B_ε , $\varepsilon \in \Omega$ are nonempty compact convex sets, we have that

$$B_{\varepsilon_0} = \cap_{\varepsilon \in \Omega} B_\varepsilon$$

is a nonempty compact convex set.

Let us suppose $\varepsilon_0 = 0$. Then $B_0 = \cap_{i \in N} A^i \neq \emptyset$ and for $v \in B_0$ we would have $\text{co}\{S^i, v\} \subset A^i$, $i \in N$ the later implying that $\cup_{i \in N} \text{co}\{S^i, v\}$ covers S and therefore also $\cup_{i \in N} A^i$ covers S , which is a contradiction. Thus $\varepsilon_0 > 0$.

We shall show first that no point of B_{ε_0} can be an interior point of some $A_{\varepsilon_0}^i$. Assuming the contrary, e.g. that $b \in B_{\varepsilon_0} \cap \text{int } A_{\varepsilon_0}^i$ we have first of all that $d(b, A^i) < \varepsilon_0$ and $d(b, A^j) \leq \varepsilon_0$, $j \in N$. Since $\cap_{j \in N \setminus \{i\}} A^j$ is nonempty, $\varepsilon_0 > 0$, the set $\cap_{j \in N \setminus \{i\}} A_{\varepsilon_0}^j$ is convex and has a nonempty interior. Now, $b \in \cap_{j \in N \setminus \{i\}} A_{\varepsilon_0}^j$ and each of its neighborhoods contains interior points of $\cap_{j \in N \setminus \{i\}} A_{\varepsilon_0}^j$. Hence so does $\text{int } A_{\varepsilon_0}^i$. Let be x a such point. Then $d(x, A^j) < \varepsilon_0$, $j \in N$. Denote by $\delta = \sup\{d(x, A^j) : j \in N\}$. It follows that $x \in B_\delta$ with $\delta < \varepsilon_0$, in contradiction with the definition of ε_0 .

Thus B_{ε_0} is on the boundary of every $A_{\varepsilon_0}^i$. Hence:

$$d(b, A^j) = \varepsilon_0, \forall j \in N \forall b \in B_{\varepsilon_0}.$$

If B_{ε_0} would contain two distinct points, b_1 and b_2 , the line segment determined by these two points would be in this set too.

The line determined by these points should meet the boundary of S which is in $\cup_{j \in N} A^j$. Thus the line would meet some set A^i in a point a . Suppose that b_1 is between a and b_2 . Let c be the point in A^i at distance ε_0 from b_2 . Consider the plane of dimension two determined by the line cb_2 and the line b_1b_2 . This plane meets the supporting hyperplane to A^i at c and perpendicular on cb_2 in a line λ which is perpendicular to cb_2 . Now, a must be behind the supporting hyperplane, hence the line b_2b_1 meets the line λ in a point d between a and b_2 . Thus the triangle $dc b_2$ is rectangular at c . Since B_{ε_0} is convex, we can suppose without loss of generality that b_1 is on the segment fb_2 , where f is the base of the perpendicular from c to b_1b_2 . But then the distance from b_1 to c is less then the distance of b_2 to c which is ε_0 . This contradiction shows that B_{ε_0} reduces to a point.

Let us observe now, that $\cup_{i \in N} A_{\varepsilon_0}^i = S$ since for $v \in B_{\varepsilon_0}$ the simplexes $\text{co}\{v, a_j : j \in N \setminus \{i\}\} \subset A^i$ form a simplicial subdivision of S .

Take $u \in S$, $u \neq v$. Then $u \notin B_{\varepsilon_0}$, and hence u must be in some $A_{\varepsilon_0}^k$ and thus $d(u, A^k) \leq \varepsilon_0$ and u will be outside of some $A_{\varepsilon_0}^j$, hence $d(u, A^j) > \varepsilon_0$. Therefore u cannot be equally spaced to every A^i , which shows that v is the single point with this property.

The same reasoning shows that the equality in the point 4. of the lemma takes place.

Let be A^1, A^2, \dots, A^{n+1} and D^1, D^2, \dots, D^{n+1} be the \mathcal{H} -families in the point 5. Then we have

$$A_{\varepsilon_0}^i \subset D_{\varepsilon_0}^i, \quad i \in N$$

with ε_0 defined at 1. Hence

$$\cap_{i \in N} A_{\varepsilon_0}^i \subset \cap_{i \in N} D_{\varepsilon_0}^i \neq \emptyset,$$

and then

$$\varepsilon_1 = \inf\{\varepsilon : \cap_{i \in N} D_{\varepsilon}^i \neq \emptyset\} \leq \varepsilon_0.$$

■

Lemma 2 *If the n -simplex S is covered by $n+1$ closed convex sets C^1, C^2, \dots, C^{n+1} such that $S^i \subset C^i$ for each $i \in N$, then putting*

$$C_t^i = (1-t)S^i + tC^i, \quad t \in [0, 1], \quad i \in N,$$

and $\mathcal{H}_t = \{C_t^i : i \in N\}$, we have the following assertions fulfilled:

1. There exists a $t_0 \in (0, 1]$ such that
 - (i) $\{C_t^i : i \in N\}$ is an \mathcal{H} -family for $t < t_0$;
 - (ii) $\{C_t^i : i \in N\}$ covers S for $t \geq t_0$.
2. If ε_t denotes the distance of the equally spaced point v_t from the members C_t^i of the \mathcal{H} -family $\{C_t^i : i \in N\}$, then ε_t is decreasing with t .
3. If $\{C_t^i : i \in N\}$ is an \mathcal{H} -family, then there exists a neighborhood W of t such that $\{C_{t'}^i : i \in N\}$ is an \mathcal{H} -family for any $t' \in W$.
4. $\delta_0 = \inf\{\varepsilon_t : \mathcal{H}_t \text{ is an } \mathcal{H} - \text{family}\} = 0$ and there exists a sequence of ε_t -s converging to 0.

Proof. Denoting with $\|\cdot\|$ the Euclidean norm in \mathbb{R}^n we have for $t \in [0, 1]$ and $x = (1-t)s + tc \in C_t^i$, ($s \in S^i$, $c \in C^i$) that $\|x - s\| = \|(1-t)s + tc - s\| = t\|c - s\| \leq td$ with d the diameter of S .

Denote with b the barycenter of S and with δ the minimal distance of b from the faces S^i . Consider an arbitrary $x \in C_t^i$ represented in the above form. Then

$$\|x - b\| \geq \|b - s\| - \|s - x\| \geq \delta - td,$$

whereby $\|x - b\| > 0$ for $t > 0$ sufficiently small. Varying i , we can get a positive t which is of this property for all $i \in N$. Taking such a t , we have $b \notin C_t^i$, $i \in N$. Thus the later sets form an \mathcal{H} -family.

We have $C_{t_1}^i \subset C_{t_2}^i$ for $t_1 < t_2$.

Indeed, for an arbitrary $(1 - t_1)s + t_1c \in C_{t_1}^i$ we have

$$(1 - t_1)s + t_1c = (1 - t_2)s + t_2\left(\frac{t_2 - t_1}{t_2}s + \frac{t_1}{t_2}c\right) \in C_{t_2}^i$$

since $S^i \subset C^i$.

Hence if $\{C_t^i : i \in N\}$ is an \mathcal{H} -family, then $\{C_{t'}^i : i \in N\}$ has for $t' < t$ the same property.

From the considerations above and the fact that $\{C_1^i : i \in N\}$ covers S follows the existence of a $t_0 \in (0, 1]$ satisfying the requirements in point 1 of the lemma.

From the fact that we have $C_{t_1}^i \subset C_{t_2}^i$ for $t_1 < t_2$ and the assertion 5 in Lemma 1, it follows the assertion 2 of the lemma.

Suppose that $\{C_t^i : i \in N\}$ is an \mathcal{H} -family. Fix i for the moment and take $x = (1 - t')s + t'c \in C_{t'}^i$. Let be v_t the point in S at the distance ε_t from the sets C_t^i . Then

$$\|x - v_t\| = \|(1 - t)s + tc + (t - t')(s - c) - v_t\| \geq \|(1 - t)s + tc - v_t\| - |t - t'|\|s - c\|.$$

The obtained relation shows that for $|t - t'|$ sufficiently small the distance of v_t from all the members of $\{C_{t'}^i : i \in N\}$ is positive, which concludes the proof of the point 3 of the lemma.

To prove the assertion 4 of the lemma, we assume the contrary: $\delta_0 > 0$.

Consider the sequence (t_m) , $t_m < t_0$, $t_m \rightarrow t_0$. For every m we determine the equally spaced point $v_m = v_{t_m}$ to the sets $C_{t_m}^i$ and its distance $\varepsilon_m = \varepsilon_{t_m}$ to these sets. Passing to a subsequence if necessary, we can suppose that $v_m \rightarrow v \in S$. We shall show, that

$$\|x - v\| \geq \delta_0 \quad \forall x \in C_{t_0}^i, \quad \forall i \in N.$$

Indeed, fixing i for the moment and taking an arbitrary $x = (1 - t_0)s + t_0c$ ($s \in S^i$, $c \in C^i$), we have

$$\|(1 - t_m)s + t_m c - v\| \geq \|(1 - t_m)s + t_m c - v_m\| - \|v_m - v\| \geq \varepsilon_m - \|v_m - v\|.$$

By passing to limit with m we conclude

$$\|x - v\| = \|(1 - t_0)s + t_0 c - v\| \geq \delta_0.$$

Hence we have for $i \in N$ that

$$d(v, C_{t_0}^i) \geq \delta_0.$$

The obtained relations show that $\{C_{t_0}^i : i \in N\}$ is an \mathcal{H} -family, which contradicts the definition of t_0 . From the point 2 and the proof above it follows also that the sequence (ε_m) decreases to 0 with $m \rightarrow \infty$. ■

Proof Theorem 1.

The proof of the first part of the theorem and the unicity of the equally spaced point when the distance is positive follows from Lemma 1.

Suppose that the family $\{A^i : i \in N\}$ covers S . Then considering the sets

$$A_t^i = (1 - t)S^i + tA^i, \quad t \in [0, 1]$$

we can determine according to Lemma 2 the maximal number $t_0 \in (0, 1]$ such that $A_t^i : i \in N$ form an \mathcal{H} -family for $t < t_0$ with the equally spaced point v_t from its members and the distance ε_t . Then we can determine a sequence of distances ε_{t_m} tending to 0 with m . Consider the point $x_m^i \in A_{t_m}^i \subset A^i$ of the distance ε_{t_m} from v_{t_m} . Fix $j \in N$. Passing if necessary to a subsequence we can suppose that $x_m^j \rightarrow x \in A^j$. Since $\varepsilon_{t_m} \rightarrow 0$ it follows that $x_m^i \rightarrow x$, $\forall i \in N$. Hence $x \in \bigcap_{i \in N} A^i$ and thus $d(x, A^i) = 0$, $i \in N$. This completes the proof of the theorem.

Proof of Corollary 1

Since every S^i is in some B^α there must exist a subfamily

$$\{B^{\alpha_1}, B^{\alpha_2}, \dots, B^{\alpha_{n+1}}\}$$

so as to have $S^i \subset B^{\alpha_i}$. Obviously $\{B^{\alpha_1}, B^{\alpha_2}, \dots, B^{\alpha_{n+1}}\}$ cannot cover S , and hence according the first part of Theorem 1, there exists a unique $v \in S$ which is at the distance $\varepsilon > 0$ to every B^{α_i} . Taking now an arbitrary other B^α , according the condition of the corollary, v must be at the same distance ε to it.

Proof of Corollary 2.

Let $\mathcal{B} = \{B^\alpha : \alpha \in I\}$ be a face covering family of closed convex sets in S . If the cardinality of I is $\leq n$, then we have nothing to prove since any $k \leq n$ maximal faces of S have a common point which will be a common point for \mathcal{B} . Hence we can suppose that \mathcal{B} contain at least $n + 1$ members.

Denote then in the following with N the set $N = \{1, 2, \dots, n + 1\}$. Let now $\mathcal{B} = \{B^\alpha : \alpha \in I\}$ be a face covering family of closed convex sets in S with cardinality of $I \geq n + 1$. Let us suppose that a subfamily $\mathcal{B}_0 = \{B^{\alpha_1}, B^{\alpha_2}, \dots, B^{\alpha_{n+1}}\}$ has a nonempty intersection and that \mathcal{B}_0 covers the boundary of S . Consider then a point $x \in \bigcap_{i \in N} B^{\alpha_i}$ and v an arbitrary point in the simplex S . The halfline with the origin in the point x and going through the point v must then intersect the boundary of the simplex S in a point y . Because the subfamily \mathcal{B}_0 is covering the boundary of S there is an $i \in N$ such that $y \in B^{\alpha_i}$. We have also $x \in B^{\alpha_i}$. From the convexity of the set B^{α_i} follows then $v \in B^{\alpha_i}$. Therefore the simplex S is also covered by the subfamily \mathcal{B}_0 .

Consider an arbitrary subfamily $\mathcal{B}_0 = \{B^{\alpha_1}, B^{\alpha_2}, \dots, B^{\alpha_{n+1}}\}$ of \mathcal{B} . If some S^i is not covered by it, then the vertex a_i which is element of every S^j , $j \neq i$ must be element of each member of the family.

Suppose that the subfamily covers the boundary of S . If some B^{α_i} contains two different maximal faces of S , then by convexity it covers S and then it contains the vertex a_k contained in the intersection of the sets B^{α_j} , $j \neq i$.

If no B^{α_i} contain two different maximal faces, then we can consider that $S^i \subset B^{\alpha_i}$, $i \in N$, and as soon by hypothesis the family \mathcal{B}_0 covers S , we have according Theorem 1 that it has a nonempty intersection.

In conclusion, each family of $n + 1$ members of \mathcal{B} has a nonempty intersection. Hence from the theorem of Helly, the whole \mathcal{B} has nonempty intersection.

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